J. Fluid Mech. (1979), *vol.* 89, *part* 4, *pp.* 785–792
Printed in Great Britain

Propagation of hydromagnetic waves in a perfectly conducting non-isothermal atmosphere in the presence of rotation and a variable magnetic field

By N. RUDRAIAH AND M. VENKATACHALAPPA

Department of Mathematics, Central College, Bangalore University, Bangalore-560001, India

(Received 20 March 1978)

The propagation of internal Alfvén–inertio–acoustic gravity waves in a perfectly electrically conducting, stratified, inviscid, non-isothermal, rotating atmosphere permeated by a non-uniform magnetic field is investigated. These waves exhibit singular properties at the critical levels at which the magnetic field and the sound velocity are such that

$$(\omega^2-S^2)\{(c^2+V^2)\,\omega^2-c^2S^2\}-(c^2+V^2)\,\bar{R}^2=0,$$

where ω is the frequency of the waves, $S=kV_x+lV_y$, $\bar{R}=2\Omega_z\omega$, V_x and V_y are the x and y components of the Alfvén velocity, k and l are the corresponding wavenumbers and c is the sonic velocity. These levels act like valves which permit waves to penetrate them from one side only and absorb them when they propagate from the other side. In contrast to the incompressible results of Acheson (1972), we show that the valve effect in compressible flow no longer requires the presence of non-zero components of rotation in the plane normal to the direction in which the medium varies. We find that the compressibility increases the probability of a valve effect and so increases the capacity of a hydromagnetic wave to propagate across a field line, rather than being absorbed at some critical level.

1. Introduction

The systematic study of internal waves (Gossard & Hooke 1975; Hines 1974) in conducting or non-conducting fluids with and without rotation in which either the background velocity or magnetic field varies with height constitutes a comparatively recent development in theoretical fluid mechanics with applications in astrophysics and geophysics. The principal aim of such a study is to establish the behaviour of waves near critical levels (Booker & Bretherton 1967; Hines 1968; Rudraiah & Venkatachalappa 1972*a*,*b*,*c*, 1974; Rudraiah, Venkatachalappa & Kandaswamy 1976, 1977; McKenzie 1973; Grimshaw 1975) and in particular to establish whether or not there is wave absorption and whether or not there is a valve effect (Acheson 1972; Eltayeb 1977).

In hydrodynamics, Bretherton (1966) and Booker & Bretherton (1967) were the first to investigate this type of problem. They studied the propagation of these waves in an incompressible, Boussinesq, inviscid, adiabatic fluid in the absence of a Coriolis force. Later Jones (1967) investigated this problem in the presence of a Coriolis force for the case in which the mean horizontal velocity $U(z)$ varies with height (z)

only. These studies reveal that as the waves propagate across certain levels, called critical levels, in the fluid they are attenuated and the wave energy is transferred to the mean flow at these levels. But in addition to internal gravity waves it is also important to consider Alfvén waves in the study of the ionosphere, the earth's core and the exosphere of the sun. Recently, Rudraiah & Venkatachalappa (1972*a*,*b*,*c*) have accounted for electromagnetic effects in an investigation of the propagation of these waves in a non-dissipative, stratified, perfectly conducting shear flow in the presence of a uniform magnetic field with and without rotation. They observed that the wave energy is transferred to the mean flow at the critical levels, called magnetic critical levels. These critical levels exist because of the shear flow, whereas Acheson (1972) has shown that the phenomenon of critical-layer absorption does not depend crucially on the presence of a mean shear flow. A non-uniform magnetic field permeating the fluid is also capable of giving rise to critical-layer absorption. In particular, Acheson (1972) has shown that a valve effect is caused by the presence of non-zero components of rotation in the plane normal to the direction in which the medium varies. Recently Eltayeb (1977) has generalized these results to linear wave motions in magnetic and/or velocity shear by employing a quantity A which is a measure of the intensity of the wave.

The work mentioned so far is concerned with the critical-layer absorption in an incompressible Boussinesq fluid. This assumption of incompressibility is valid only when the speed of flow is much less than that of sound in the medium. But the conditions under which internal Alfvén–acoustic–gravity waves are important in geophysics and astrophysics are usually far removed from this idealization. In such circumstances, we have to consider the effect of compressibility on the propagation of internal waves. McKenzie (1973) has discussed the general nature of critical levels for any type of wave propagation in a stratified, incompressible or compressible medium and has shown that a critical level at which a wave packet is neither reflected nor transmitted can exist only if the wave normal curve possesses an asymptote which is parallel to the direction of the variation of the properties of the medium through which the wave propagates. Later, Rudraiah *et al.* (1976, 1977) considered the effect of compressibility on the propagation of internal Alfvén–acoustic–gravity waves under the assumption of an isothermal atmosphere and in the absence of rotation.

However, for astrophysical and geophysical applications (see Acheson & Hide 1973) one has to consider the propagation of internal Alfvén–inertio–acoustic–gravity waves in a non-isothermal rotating conducting fluid permeated by a variable magnetic field with the object of understanding the distortion of the critical levels due to the combined effect of rotation and a variable magnetic field permeating a non-isothermal atmosphere. This distortion will provide an adequate explanation for the distribution of ionization in the upper atmosphere. Therefore the aim of the present paper is to consider the effect of rotation on the propagation of hydromagnetic internal gravity waves in an inviscid, perfectly conducting, non-isothermal, compressible fluid in the presence of a non-uniform magnetic field. In other words this paper is an extension of the incompressible analysis of Acheson (1972) to compressible fluid, the essential difference being that the valve effect no longer requires the presence of non-zero components of rotation in the plane normal to the direction in which the medium varies.

It is shown that the compressibility increases the probability of a valve effect and so increases the capacity of a hydromagnetic wave to propagate across a field line, rather than being absorbed at some critical level; this is of importance in geophysics.

2. Mathematical formulation

We consider a system of Cartesian axes with the z axis in the vertical direction. We assume as a model a non-isothermal compressible ideal fluid with vertical density stratification. This model is assumed to rotate with an angular velocity

$$\mathbf{\Omega} = (\Omega_x, \Omega_y, \Omega_z).$$

Under these assumptions the basic magnetohydrodynamic equations for this model are

$$\rho_1\left[\frac{\partial \mathbf{q}}{\partial t}+(\mathbf{q}\,.\nabla)\,\mathbf{q}+2\mathbf{\Omega}\times\mathbf{q}\right] = -\nabla p_1+\mathbf{g}\rho_1+\mu_m(\nabla\times\mathbf{H})\times\mathbf{H}, \tag{2.1}$$

$$\partial\rho_1/\partial t+\nabla\,.(\rho_1\mathbf{q}) = 0, \tag{2.2}$$

$$\partial p_1/\partial t+(\mathbf{q}\,.\nabla)\,p_1 = c^2[\partial\rho_1/\partial t+(\mathbf{q}\,.\nabla)\,\rho_1], \tag{2.3}$$

$$\partial\mathbf{H}/\partial t = \nabla\times(\mathbf{q}\times\mathbf{H}) \tag{2.4}$$

and

$$\nabla\,.\mathbf{H} = 0, \tag{2.5}$$

where $\mathbf{q}$ denotes the flow velocity, ρ_1 the density, p_1 the hydrodynamic pressure, c the sound speed, which varies with the height z, $\mathbf{H}$ the magnetic field and μ_m the magnetic permeability.

2.1. *Equilibrium configuration*

We assume that a non-uniform basic magnetic field $\mathbf{H}_0(z)$ permeates the entire compressible perfectly conducting fluid and that the fluid is in rigid-body rotation with an angular velocity $\mathbf{\Omega} = (\Omega_x, \Omega_y, \Omega_z)$. We also assume that the density of the compressible fluid is stratified in the z direction, i.e. $\rho_1 = \rho_0(z)$. To be consistent with the equations of motion, we assume, following Acheson (1972), a more general basic magnetic field $\mathbf{H}_0(z)$ of the form

$$\mathbf{H}_0(z) = \{H_x(z), H_y(z), 0\}, \tag{2.6}$$

where $H_x(z)$ and $H_y(z)$ are arbitrary functions of z. For magnetostatic balance we have

$$\frac{dp_0}{dz} = -\left(g\rho_0+\mu_m H_x\frac{dH_x}{dz}+\mu_m H_y\frac{dH_y}{dz}\right), \tag{2.7}$$

where p_0 denotes the equilibrium hydrodynamic pressure. From the equation for the speed of sound,

$$c^2 = \gamma p_0/\rho_0, \tag{2.8}$$

and using (2.7), we find that the density stratification $\rho_0(z)$ is of the form

$$\rho_0(z) = \rho_c \exp\left(-\int_0^z \beta(z)\,dz\right),$$

where ρ_c is the reference density at $z = 0$,

$$\beta = (1+\dot{H})/H, \quad H(z) = (c^2+\tfrac{1}{2}\gamma V^2)/(g\gamma), \tag{2.9}$$

$V^2 = \mu_m H_0^2/\rho_0$ is the local Alfvén velocity and γ is the usual ratio of specific heats.

2.2. *The perturbed state*

On the equilibrium configuration discussed above we superpose a small disturbance of the form

$$(u, v, w), \quad \rho_0+\rho, \quad p_0+p, \quad (H_x(z)+h_x,\ H_y(z)+h_y,\ h_z).$$

We assume that the disturbances are sufficiently small compared with the initial state that the higher-order terms in the perturbed quantities can be neglected. Equations (2.1)–(2.5) then reduce to a set of linear partial differential equations in which all perturbation quantities f may be written as

$$f = \mathrm{Re}\,[\hat{f}(z)\exp\{i(kx+ly-\omega t)\}]. \tag{2.10}$$

These linear differential equations can be combined to yield the wave equation in w, which on using the substitution

$$\hat{w}(z) = \tilde{w}(z)\exp\left(\int_0^z \tfrac{1}{2}\beta dz\right) \tag{2.11}$$

becomes

$$\begin{aligned} R\frac{d^2\tilde{w}}{dz^2} &+ \left[\frac{dR}{dz}-\frac{R}{P}\frac{dP}{dz}-2i\bar{R}\{2c^2T\omega-\bar{S}(2\bar{L}\omega-gS)\}\right]\frac{d\tilde{w}}{dz}+\Big[4c^2T^2\omega^2+P(\omega^2-S^2-N^2)\\ &+N^2\omega^2(\omega^2-S^2-4\Omega_z^2)-4\Omega^4\bar{\Omega}^2-g\alpha^2(\omega^2-S^2)-4\alpha^2\omega^2\bar{L}^2+4g\omega\bar{T}-4g\alpha^2\omega S\bar{L}\\ &+\tfrac{1}{2}R(-\tfrac{1}{2}\beta^2+\beta')+\tfrac{1}{2}\beta\left(\frac{dR}{dz}-\frac{R}{P}\frac{dP}{dz}\right)-2\beta\omega c^2\bar{T}(\omega^2-S^2)-2\beta\omega^3L\bar{S}-\frac{1}{P}\frac{dP}{dz}\\ &\times\{-g\omega^2(\omega^2-S^2+\bar{R})-2\bar{R}T\omega ic^2+2\omega c^2\bar{T}(\omega^2-S^2)-g\bar{R}iS\bar{S}+2\omega^3L\bar{S}+2i\bar{R}\omega\bar{L}\bar{S}\}\\ &+(g\omega^2-2c^2\omega\bar{T})\frac{dS^2}{dz}-g\bar{R}i\frac{d(S\bar{S})}{dz}+2\omega^3\frac{d(L\bar{S})}{dz}+2i\omega\bar{R}\frac{d(\bar{L}\bar{S})}{dz}\\ &+2\frac{dc^2}{dz}\{-\omega^2g\alpha^2-2\bar{R}T\omega+2\omega\bar{T}(\omega^2-S^2)\}\Big]\tilde{w} = 0, \end{aligned} \tag{2.12}$$

where

$$\begin{aligned} R &= (\omega^2-S^2)\{(c^2+V^2)\omega^2-c^2S^2\}-(c^2+V^2)\bar{R}^2,\\ P &= \omega^2(\omega^2-\alpha^2V^2-\bar{R})-\alpha^2c^2(\omega^2-S^2),\\ L &= \Omega_xV_x+\Omega_yV_y, \quad \bar{L} = \Omega_xV_y-\Omega_yV_x, \quad N^2 = g\beta,\\ S &= kV_x+lV_y, \quad \bar{S} = lV_x-kV_y,\\ T &= k\Omega_x+l\Omega_y, \quad \bar{T} = l\Omega_x-k\Omega_y,\\ \bar{R} &= 2\Omega_z\omega, \quad \alpha^2 = k^2+l^2,\\ V^2 &= V_x^2+V_y^2, \quad V_x^2 = \mu_mH_x^2/\rho_0, \quad V_y^2 = \mu_mH_y^2/\rho_0 \end{aligned}$$

and

$$\bar{\Omega}^2 = \Omega_x^2+\Omega_y^2.$$

We note that the governing wave equation (2.12) is singular at heights at which the Alfvén velocities V_x and V_y and the sound speed c are such that $R(z) = 0$ or $P(z) = 0$. However, when V_x, V_y and c are uniform the coefficients of (2.12) are constants and the equation has the solution $\tilde{w} \propto \exp(imz)$, where m is a constant vertical wave-number. The main aim of this paper is to investigate the propagation of Alfvén–acoustic–gravity waves in the presence of a variable magnetic field $\{H_x(z),\ H_y(z),\ 0\}$

and variable sound speed $c(z)$ in a fluid with density stratification $\rho_0(z)$. Since the most striking feature of (2.12) is its singularities, we discuss below the propagation of waves in the neighbourhood of these singularities.

3. Wave propagation in the neighbourhood of singular levels

In this section we discuss the solution of the wave equation (2.12) near the singular levels, namely $R(z) = 0$. Using the method of Frobenius, we find that the solution of (2.12) valid in the neighbourhood of $R(z) = 0$ is

$$\tilde{w}(z) = A[1+a_1(z-z_c)+a_2(z-z_c)^2+\ldots] \\ +B(z-z_c)^{2i\mu}[1+b_1(z-z_c)+b_2(z-z_c)^2+\ldots], \tag{3.1}$$

where z_c is such that $R(z_c) = 0$,

$$\mu = \left[\frac{\bar{R}}{dR/dz}\{2c^2T\omega+\bar{S}(gS-2\bar{L}\omega)\}\right]_{z=z_c}, \tag{3.2}$$

$a_1, a_2, \ldots, b_1, b_2 \ldots$ are known constants and A and B are constants of integration. We note that, in the limit $c^2 \to \infty$ and $dc/dz \to 0$, μ reduces to

$$\mu = \frac{\omega(k\Omega_x+l\Omega_y)}{dS^2/dz}\operatorname{sgn}\{(S^2-\omega^2)\bar{R}\}, \tag{3.3}$$

which is the same as the formula obtained by Acheson (1972) in the case of incompressible perfectly conducting fluid. We shall first consider the second solution in (3.1). Following the analysis of Booker & Bretherton (1967) and Acheson (1972) we find that, if $\omega = \omega_r + i\omega_i$ with $\omega_i > 0$, the solution of (2.12) will be similar to (3.1) with z_c replaced by

$$z_c + \frac{2i\omega_r\omega_i}{dR/dz}(c^2+V^2)\,\Gamma,$$

where

$$\Gamma = 4\Omega_z^2 + \frac{2+M^2}{1+M^2}S^2 - 2\omega_r^2$$

and $M = V/c$ is the magnetic Mach number. Finally letting $\omega_i \to 0$, we obtain the matching condition that if

$$\left.\begin{aligned} &(z-z_c)^{2i\mu} = \exp\{2i\mu\log(z-z_c)\} \quad \text{for} \quad z > z_c \\ \text{then} \quad &(z-z_c)^{2i\mu} = \exp\{2i\mu\log|z-z_c|\}\exp(2|\mu|\pi) \quad \text{for} \quad z < z_c, \end{aligned}\right\} \tag{3.4}$$

where we have assumed for the sake of definiteness that $\mu\omega\Gamma > 0$. Thus the magnitude of the second term in (3.1) is not the same on the two sides of the critical level but differs by a factor of $\exp(2|\mu|\pi)$. In other words, the amplitudes of the wave on the two sides of the critical level differ by a factor of $\exp(2|\mu|\pi)$. This difference in amplitude above and below the critical level can be interpreted physically by knowing whether the wave is propagating upwards or downwards. The interpretation of upward or downward propagation of waves can be given by considering the transport of energy due to wave motion.

In the presence of the magnetic field, the total mean rate of the work done by the conducting compressible fluid below any level on the fluid above is $\overline{p_T w}$, where p_T

is the total pressure due to the hydrodynamic and hydromagnetic pressures and the overbar denotes a time average. We have

$$\overline{p_T w} = \overline{\{p+\mu_m H_x h_x+\mu_m H_y h_y\} w}.$$

Substituting for h_x and h_y in terms of w, which can be obtained from the linearized equations of motion, we obtain

$$\overline{p_T w} = \frac{\rho_c}{2P}\,\mathrm{Re}\left[\frac{R}{\omega}\,i\,\frac{d\tilde{w}}{dz}\,\tilde{w}^* + 2\Omega_z\{\bar{S}(gS-2\omega\bar{L})+2\omega c^2 T\}\tilde{w}\tilde{w}^*\right], \tag{3.5}$$

where $\tilde{w}^*$ is the complex conjugate of $\tilde{w}$. By differentiating (3.5) with respect to z and using the wave equation (2.12), we get

$$d(\overline{p_T w})/dz = 0. \tag{3.6}$$

Hence the total upward energy flux is conserved everywhere except at the critical level, where the substitution of (2.12) is invalid. Hence we can take the energy flux $\overline{p_T w}$ as the measure of the strength of the wave.

For the second solution in (3.1) we find that the energy flux is given by

$$\overline{p_T w} = \begin{cases} -\dfrac{\rho_c\mu}{2\omega P}\left(\dfrac{dR}{dz}\right)_{z=z_c} |B|^2 \quad \text{for} \quad z > z_c, \\[2ex] -\dfrac{\rho_c\mu}{2\omega P}\left(\dfrac{dR}{dz}\right)_{z=z_c} |B|^2 \exp(4|\mu|\pi) \quad \text{for} \quad z < z_c. \end{cases} \tag{3.7}$$

If $\mu\omega P dR/dz < 0$, $\overline{p_T w}$ is always positive. In other words energy is flowing upwards. Thus the second solution can be interpreted as an upward-propagating wave. Thus from (3.7) we find that as the wave propagates through the critical level it is attenuated. If $\mu\omega P dR/dz > 0$, $\overline{p_T w}$ is always negative and the wave is again attenuated by a factor $\exp(4|\mu|\pi)$. The energy flux in the neighbourhood of the critical level associated with the first solution in (3.1) is

$$\overline{p_T w} = \frac{\rho_c \mu}{2\omega P}\left(\frac{dR}{dz}\right)_{z=z_c} |A|^2 \quad \text{for} \quad z \gtrless z_c.$$

Thus, if $\mu\omega P dR/dz < 0$, $\overline{p_T w}$ is negative and hence propagates downwards and is not attenuated. If $\mu\omega P dR/dz > 0$ the first solution represents an upward-propagating unattenuated wave. Thus the direction of propagation of the wave represented by first solution in (3.1) is always opposite to that represented by the second solution and a wave crossing its critical level $z = z_c$ at which $R(z_c) = 0$ will emerge with or without attenuation according as

$$W\mu\omega P\,dR/dz \gtrless 0,$$

where W is its velocity of propagation in the z direction. Thus, even in the case of non-isothermal compressible fluid, we observe the valve effect found by Acheson (1972) in an incompressible fluid. This valve effect exists because of the non-zero value of μ. It is very important to compare the results of the present analysis with the incompressible results of Acheson in order to obtain the effect of compressibility on the valve effect. Therefore, from the incompressible result of Acheson (1972), namely (3.3), we find that the valve effect exists because of the presence of non-zero components Ω_x and Ω_y of rotation in the plane normal to the direction in which the medium varies, whereas the value of μ given by (3.2) shows that the valve effect in compressible fluid no longer requires the presence of the above non-zero components of rotation. In other words, compressibility increases the probability of a valve effect and hence increases the capacity of a hydromagnetic wave to propagate across a field line rather than being absorbed at some critical level.

The above results for the valve effect can also be obtained from the group velocity near the critical level as discussed for non-isothermal flow by Rudraiah *et al.* (1976). We note that the critical level $P(z) = 0$ is not significant in discussing the attenuation of waves since the corresponding solution does not represent attenuation of waves and hence it is omitted from discussions.

4. Conclusions

It is shown that the governing wave equation is singular at the height z_c where the magnetic field $H(z_c)$ and the sound velocity $c(z_c)$ reach critical values such that

$$\{\omega^2 - S(z_c)\}[\{c^2(z_c) + V^2(z_c)\}\omega^2 - c^2(z_c)S^2(z_c)] - \{c^2(z_c) + V^2(z_c)\}\bar{R}^2 = 0.$$

Since the total energy flux across the field lines is constant, we find that it is an appropriate measure of the magnitude of the waves. The propagation of waves near the critical levels is discussed using the transport of energy due to wave motions. We find that a hydromagnetic wave in a rotating non-isothermal compressible fluid approaching its critical level from one side will be highly attenuated, but if it approaches from the other side will be transmitted without attenuation. Comparing our compressible results with the incompressible results of Acheson (1972), we conclude that the valve effect exists even in the absence of components of rotation in the plane normal to the direction in which the medium varies. From this we find that the compressibility increases the probability of a valve effect and so increases the capacity of a hydromagnetic wave to propagate across a field line rather than being absorbed at some critical level.

The authors are grateful to the referee for many valuable suggestions which have improved the paper significantly. This work is supported by the University Grants Commission of India under research project F-23-237/75 SR-II.

REFERENCES

Acheson, D. J. 1972 *J. Fluid Mech.* **53**, 401.
Acheson, D. J. & Hide, R. 1973 *Rep. Prog. Phys.* **36**, 159.
Booker, J. R. & Bretherton, F. P. 1967 *J. Fluid Mech.* **27**, 513.
Bretherton, F. P. 1966 *Quart. J. Roy. Met. Soc.* **92**, 466.
Eltayeb, I. A. 1977 *Phil. Trans. Roy. Soc.* A **285**, 607.
Grimshaw, R. 1975 *J. Fluid Mech.* **70**, 287.
Gossard, E. E. & Hooke, W. H. 1975 *Waves in the Atmosphere.* Elsevier.
Hines, C. O. 1968 *J. Atmos. Terr. Phys.* **30**, 837.
Hines, C. O. 1974 *The Upper Atmosphere in Motion.* Washington: Am. Geophys. Un.

Jones, W. L. 1967 *J. Fluid Mech.* **30**, 439.
McKenzie, J. F. 1973 *J. Fluid Mech.* **58**, 709.
Rudraiah, N. & Venkatachalappa, M. 1972*a* *J. Fluid Mech.* **52**, 1093.
Rudraiah, N. & Venkatachalappa, M. 1972*b* *J. Fluid Mech.* **54**, 209.
Rudraiah, N. & Venkatachalappa, M. 1972*c* *J. Fluid Mech.* **54**, 217.
Rudraiah, N. & Venkatachalappa, M. 1974 *J. Fluid Mech.* **62**, 705.
Rudraiah, N., Venkatachalappa, M. & Kandaswamy, P. 1976 *J. Fluid Mech.* **73**, 125.
Rudraiah, N., Venkatachalappa, M. & Kandaswamy, P. 1977 *J. Fluid Mech.* **80**, 223.